J. Fluid Mech. (1992), vol. 238, pp. 579–598
Printed in Great Britain

Motion of a rigid particle in Stokes flow: a new second-kind boundary-integral equation formulation

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(Received 6 August 1990 and in revised form 29 October 1991)

A new singular boundary-integral equation of the second kind is presented for the stresses on a rigid particle in motion in Stokes flow. The integral equation is particularly suitable for the mobility problem – when the forces and moments on the particle are given. A generalized Faxén law is also presented. The power of the method is demonstrated by easily reproducing known results as well as new ones, both analytically and numerically, in infinite medium as well as in confined regions.

1. Introduction

Motion of small particles in an incompressible viscous fluid occurs in many areas of engineering, e.g. sedimentation problems, lubrication processes, locomotion of flagella, etc. All these motions are characterized by low Reynolds numbers and are described by the solution of the Stokes equations.

Usually, the solutions of the Stokes flows are very specific, i.e. one solves for a specific shape of body in a given flow regime, and considerable efforts are invested in studying each new case (Ganatos, Weinbaum & Pfeffer 1980*a*; Ganatos, Pfeffer & Weinbaum 1980*b*). One way around this difficulty is to apply the singularity method: singular forces are distributed on the axis of the body or on its surface and the velocity that they induce is equated to the given velocity, thus yielding an integral equation for the singularity strengths. For simple cases like slender bodies (Johnson & Wu 1979; Liron & Mochon 1976; Liron 1978, 1984; Barta & Liron 1988*a*, *b*) or for an ellipsoid of revolution (Chwang 1975; Chwang & Wu 1974, 1975) where a distribution of forces along the axis is sufficient to describe the complete motion, general rules were derived. The types of singularities that have to be used, as well as the ratio between their strengths, is known. However, most cases require distribution of forces on the surface of the body and, although the formulation of the integral equations that describe the flow has been known for many years (Ladyzhenskaya 1963), until recently no constructive and efficient way of solution was found. The reason for this is that Ladyzhenskaya formulated the solution in terms of eigenfunctions but gave no clue as to how to determine these eigenfunctions. Others have tried to handle the problem without the eigenfunctions but this implies solution of Fredholm equations of the first kind, which is known to be an ill-posed problem. Nevertheless, people have solved such problems, starting with Youngren & Acrivos (1975), and recently Hsu & Ganatos (1989), and also Tran-Cong & Phan-Thien (1989). A much more extensive review of previous work may be found in the Introduction to Hsu & Ganatos' recent paper.

There have been many papers on particle movement in Stokes flow. For an extensive review of numerical methods in Stokes flow, the reader is referred to the recent survey by Weinbaum, Ganatos & Yan (1990), and to the paper on integral equations of the second kind for Stokes flow by Karrila & Kim (1989). Their work is not covered in Weinbaum *et al.*

The search for an integral equation of the second kind was first successful when Power & Miranda (1987) gave such a representation for the case of 'resistance' problems, i.e. when the velocity of the particle is known, and the forces and moments are to be found. Power & Miranda represented the velocity as a double-layer integral, to which they added a Stokeslet and a Rotlet, both located at the centre of the body. Equating the representation to the given velocity resulted in a Fredholm integral equation of the second kind in the double-layer density, and a numerical solution became possible after relating the Stokeslet and Rotlet strengths (force and moment) to the unknown double-layer density. Karrila & Kim (1989) and Karrila, Fuentes & Kim (1989) showed the completion of the double-layer representation by Power & Miranda to be one of many possible completions. They suggested the same representation for the velocity as did Power & Miranda, and discuss various completions, suggesting one which is advantageous to an iterative numerical process for multiparticle systems. Both these completions are successful because, as observed by Power & Miranda, and previously by Ladyzhenskaya (1963), the double-layer representation alone is able to represent flow fields that correspond to the total force and total moment equal to zero. The Karrila & Kim approach yields a Fredholm integral equation of the second kind, both for the resistance problem *and* for the 'mobility problem' (when the force and moment are given for the body or bodies). We suggest here a different approach. We represent the velocity as a single layer potential, and derive a Fredholm integral equation of the second kind where the density function is the stress on the surface itself. This approach is particularly useful for the mobility problem.

In §2 we derive the new integral equations both in infinite and in confined regions. Section 3 presents the generalized Faxén law, which may be used if the motion of the particle is our only interest. In §4 we demonstrate the strength of the equation in obtaining analytic results, both well known as well as new. In §5 we demonstrate the power of the method numerically, by easily reproducing a variety of solutions previously worked out. A discussion concludes the paper.

2. Integral equation

We want to solve the problem of a rigid body or bodies S, moving under external forces in Stokes flow. The equations of flow are the Stokes equations

$$\nabla p = \mu \nabla^2 \boldsymbol{u}, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2.2}$$

for the velocity and pressure in the medium.

Define the stress exerted by the body on the fluid as

$$\boldsymbol{f} = \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}, \tag{2.3}$$

where $\boldsymbol{\sigma}(\boldsymbol{u})$ is the stress tensor due to the flow $(\boldsymbol{u}, p)$ and $\boldsymbol{n}$ is the inward normal. Then, the total force $\boldsymbol{F}$, and the total moment $\boldsymbol{M}$ on each body is given, and

$$\boldsymbol{F} = \int_{\partial S} \boldsymbol{f} \, \mathrm{d}S, \tag{2.4}$$

$$M = \int_{\partial S} r \times f \,\mathrm{d}S. \tag{2.5}$$

On all other boundaries, the velocity is given.

2.1. *Infinite medium*

We first deal with the case of infinite medium.

Assume a given Stokes flow $(\bar{u}, \bar{p})$ into which the body is inserted, inducing a velocity u. Let

$$\left.\begin{aligned} t_{lk}(x,y) &= \frac{1}{8\pi\mu}\left(\frac{\delta_{lk}}{r}+\frac{r_l r_k}{r^3}\right), \\ p_k(x,y) &= \frac{1}{4\pi}\frac{r_k}{r^3}, \end{aligned}\right\} \quad r = x-y, \quad l,k = 1,2,3, \tag{2.6}$$

be the well-known Stokeslet in infinite medium. Then the solution to (2.1), (2.2) can be written as

$$u_k(y) = \bar{u}_k(y) + \int_{\partial S} f_l(x)\, t_{lk}(x,y)\,\mathrm{d}S_x, \quad k = 1,2,3, \tag{2.7}$$

$$p(y) = \bar{p}(y) + \int_{\partial S} f_l(x)\, p_l(x,y)\,\mathrm{d}S_x; \tag{2.8}$$

where f is the stress due to the resultant field (u, p) (and not the perturbed field).

While the above representation is well known for the case $\bar{u} = 0$ it is not clear to us how well known it is for $(\bar{u}, \bar{p})$ not vanishing. Howells (1974) indicates knowledge of such a representation when dealing with the more general Brinkman's equation. Caflisch & Rubinstein (1986) also assumed this relation in treating Faxén's law. A representation close to a proof was given by Rallison & Acrivos (1978). They proved that they could write $u_k(y)$ as in (2.7) but with an additional term due to the stress of the Stokeslets, i.e.†

$$u_k(y) = \bar{u}_k(y) + \int_{\partial S} f_l(x)\, t_{lk}(x,y)\,\mathrm{d}S_x - \int_{\partial S} u_j(x)\, n_i(x)\, \sigma_{ij}(t_{\cdot k}(x,y))\,\mathrm{d}S_x. \tag{2.9}$$

From this it is easily shown that this additional term vanishes for rigid body motions, resulting in (2.7).

Durlofsky & Brady (1989) also write (2.9) and deduce (2.7) (p. 44, equation (2.7e)). They attribute (2.9) to Ladyzhenskaya, but she only treats the case $\bar{u} = 0$ in her book. Kim (1985) also derives (2.7) for the case that the particle is stationary, and finally gives a complete proof in his new book, Kim & Karrila (1991).

The above representation holds for all y, both outside and inside the body.

Using (2.7), (2.8), one obtains

$$\sigma_{ij}(y) = \bar{\sigma}_{ij}(y) + \frac{3}{4\pi}\int_{\partial S} \frac{r_i r_j r_k}{r^5} f_k(x)\,\mathrm{d}S_x, \quad r = x-y, \quad i,j = 1,2,3, \tag{2.10}$$

where σ, $\bar{\sigma}$ are the stress tensors of the fields (u, p), $(\bar{u}, \bar{p})$, respectively.

Letting y approach the boundary along a normal to the boundary, multiplying by

† We are indebted to a referee for pointing this out to us.

the inward normal and using the 'jump condition', as in Ladyzhenskaya (1963), one obtains the singular integral equation

$$\tfrac{1}{2}f_i(y) = \bar{f}_i(y) + \left(\frac{3}{4\pi}\int_{\partial S} \frac{r_i r_j r_k}{r^5} f_k(x)\,\mathrm{d}S_x\right) n_j(y), \quad i = 1,2,3, \tag{2.11}$$

since the jump condition adds $\frac{1}{2}f_i(y)$ to the integral.

An alternative form of this equation is

$$f_i(y) = \bar{f}_i(y) + \frac{3}{4\pi}\int_{\partial S} \frac{r_i r_j r_k}{r^5}\{f_k(x)\,(n_j(y)-n_j(x)) + n_j(x)\,(f_k(x)-f_k(y))\}\,\mathrm{d}S_x, \quad i = 1,2,3. \tag{2.12}$$

Equation (2.11) or (2.12) is a singular, non-homogeneous integral equation of the second kind, for the stresses exerted by the body on the fluid. The homogeneous equation is exactly the adjoint equation for the exterior problem, discussed by Ladyzhenskaya (1963, Chapter 3, equation (38)). Ladyzhenskaya proved that the homogeneous equation (2.11) has exactly six linearly independent solutions. It follows, therefore, that (2.11) or (2.12), together with the six additional conditions on the total force and total moment, equations (2.4), (2.5), uniquely determine the solution. (This is not, however, Ladyzhenskaya's second boundary-value problem, in which the stress on the body's surface is given.) A solution to (2.11) exists if and only if $\bar{\boldsymbol{f}}$ is orthogonal to the homogeneous solutions of its adjoint equation. But (see Ladyzhenskaya), these are $\boldsymbol{\varphi}^k$, $k = 1, \ldots, 6$, which are the three translations and three rotations. Thus

$$\int_{\partial S} \bar{\boldsymbol{f}}\cdot\boldsymbol{\varphi}^k = 0, \quad k = 1,2,\ldots,6,$$

since this yields the three components of the force and torque exerted by the field $\bar{\boldsymbol{u}}$ on S, and these are zero since $\bar{\boldsymbol{u}}$ is regular in S. Once this system is solved, the velocity at every point may be evaluated using (2.7).

Equations (2.11) or (2.12) are integral equations with a weak singularity. This may be seen, as suggested by Power & Miranda (1987), as follows. Define, as in Ladyzhenskaya (1963),

$$K_{ij}(x,y) = +\frac{3}{4\pi}\frac{r_i r_j r_k}{r^5} n_k(y), \quad \boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}, \quad r = |\boldsymbol{r}|, \quad i,j = 1,2,3,$$

then

$$K_{ij}(x,y) = -\frac{3}{4\pi}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial}{\partial n(y)}\left(\frac{1}{r}\right).$$

But

$$\mathrm{d}\Omega = \frac{\partial}{\partial n(y)}\left(\frac{1}{r}\right)\mathrm{d}S_y,$$

where Ω is the solid angle. Thus, with an appropriate change of variables, the integrand in (2.12) vanishes for $\boldsymbol{y} = \boldsymbol{x}$.

Power & Miranda went on to conclude that in a numerical solution they may set the integrand equal to zero, for $\boldsymbol{y} = \boldsymbol{x}$. This is incorrect unless the above transformation (change of variables) is first performed, see also §5.

2.2. *Non-infinite medium*

For the case where boundaries exist, a similar equation to (2.11) holds. Let $\bar{\boldsymbol{u}}$ be the non-perturbed flow, which satisfies the boundary conditions (adherence to walls).

Define the Green's functions:

$$T_{ik} = t_{ik} + \tau_{ik}, \quad P_k = p_k + \pi_k, \quad i, k, = 1, 2, 3, \tag{2.13}$$

where t_{ik}, p_k are defined in (2.6), and τ_{ik}, π_k, $k = 1, 2, 3$ are regular solutions of (2.1), (2.2), such that $T_{ik} = 0$, $i, k = 1, 2, 3$ on the boundaries. Then, a representation of the final velocity $\boldsymbol{u}$, after inserting the body S, can be given similar to (2.7), (2.8):

$$\left.\begin{aligned} u_k(\boldsymbol{y}) &= \bar{u}_k(\boldsymbol{y}) + \int_{\partial S} f_l(\boldsymbol{x})\, T_{lk}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}S_x, \\ p(\boldsymbol{y}) &= \bar{p}(\boldsymbol{y}) + \int_{\partial S} f_l(\boldsymbol{x}) P_l(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}S_x. \end{aligned}\right\} \quad k = 1, 2, 3, \tag{2.14}$$

Proceeding as before one obtains the equation

$$\tfrac{1}{2} f_i(\boldsymbol{y}) = \bar{f}_i(\boldsymbol{y}) + \left(\int_{\partial S} f_k(\boldsymbol{x})\, (\boldsymbol{\sigma}_y)_{ij} (T_{\cdot k})\, \mathrm{d}S_x\right) n_j(\boldsymbol{y}), \quad i = 1, 2, 3. \tag{2.15}$$

This equation is similar to (2.11), except that the kernel has an additional term, arising from the additional solution τ_{ik}, π_k. These functions, and their derivatives, are regular in the region, by definition. Again, this equation, together with conditions (2.4), (2.5), uniquely determine the solution.

Explicit solutions for a Stokeslet $\boldsymbol{T}, \boldsymbol{P}$ are known for the case where the Stokeslet is above a plane wall (Blake 1971), outside a sphere (Oseen 1927), between the two plane walls (Liron & Mochon 1976) and inside an infinite straight cylinder (Liron & Shahar 1978).

3. The general Faxén law

If one is interested only in the velocity of the solid particle and not in the resultant flow fields, general Faxén laws may be derived. We present here their full derivation since the laws are intrinsically connected with preknowledge of the solutions to (2.11) or (2.15). The derivation here follows lines similar to Brenner (1964, 1966).

Suppose we have a regular Stokes flow field $\bar{\boldsymbol{u}}$, satisfying given boundary conditions. Insert a rigid particle S, such that the final velocity is $\boldsymbol{u}$. Then on the particle

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{3.1}$$

where $\boldsymbol{V}, \boldsymbol{\Omega}$ are fixed vectors, and $\boldsymbol{u} - \bar{\boldsymbol{u}} \rightarrow 0$, when $\boldsymbol{r} \rightarrow \infty$ and on all other boundaries.

Define a *legitimate field* $\boldsymbol{v}, p$, as a flow field solving the Stokes' equations (2.1), (2.2), regular outside the particle S, and vanishing on all other boundaries (including infinity).

Take any legitimate field $\boldsymbol{v}$, and apply Green's theorem to the domain exterior to S. This yields

$$\int_{\partial S} \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{u} - \bar{\boldsymbol{u}})\, \mathrm{d}S = \int_{\partial S} (\boldsymbol{u} - \bar{\boldsymbol{u}}) \cdot \boldsymbol{f}(\boldsymbol{v})\, \mathrm{d}S, \tag{3.2}$$

where $\boldsymbol{f}$ is the stress on the boundary. Inserting (3.1) into (3.2) one obtains,

$$\begin{aligned} \int_{\partial S} \boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{u} - \bar{\boldsymbol{u}})\, \mathrm{d}S &= \int_{\partial S} (\boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r} - \bar{\boldsymbol{u}}) \cdot \boldsymbol{f}(\boldsymbol{v})\, \mathrm{d}S \\ &= \boldsymbol{V} \cdot \boldsymbol{F}(\boldsymbol{v}) + \boldsymbol{\Omega} \cdot \boldsymbol{M}(\boldsymbol{v}) - \int_{\partial S} \bar{\boldsymbol{u}} \cdot \boldsymbol{f}(\boldsymbol{v})\, \mathrm{d}S, \end{aligned} \tag{3.3}$$

where $\boldsymbol{F}, \boldsymbol{M}$ are the total force and total moment respectively, acting on the particle, as defined in (2.4), (2.5).

Define six legitimate fields $\boldsymbol{v}^j, j = 1, \ldots, 6$ by demanding the additional conditions:

$$\boldsymbol{v}^j(\partial S) = \boldsymbol{e}^j, \quad \boldsymbol{v}^{j+3}(\partial S) = \boldsymbol{e}^j \times \boldsymbol{r}, \quad j = 1,2,3, \tag{3.4}$$

in other words the flow fields due to the three translations and three rotations of the body. Then, inserting (3.4) into (3.3), one obtains

$$\boldsymbol{e}^j \cdot \int_{\partial S} [\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\bar{\boldsymbol{u}})]\,\mathrm{d}S = F_j(\boldsymbol{u}) = \boldsymbol{V}\cdot\boldsymbol{F}(\boldsymbol{v}^j) + \boldsymbol{\Omega}\cdot\boldsymbol{M}(\boldsymbol{v}^j) - \int_{\partial S} \bar{\boldsymbol{u}}\cdot\boldsymbol{f}(\boldsymbol{v}^j)\,\mathrm{d}S, \quad j = 1,2,3, \tag{3.5}$$

and similarly,

$$M_j(\boldsymbol{u}) = \boldsymbol{V}\cdot\boldsymbol{F}(\boldsymbol{v}^{j+3}) + \boldsymbol{\Omega}\cdot\boldsymbol{M}(\boldsymbol{v}^{j+3}) - \int_{\partial S} \bar{\boldsymbol{u}}\cdot\boldsymbol{f}(\boldsymbol{v}^{j+3})\,\mathrm{d}S, \quad j = 1,2,3, \tag{3.6}$$

since the total force and total moment exerted by the field $\bar{\boldsymbol{u}}$, regular in S, is zero.

The above equations are six equations for the six unknowns $\boldsymbol{V}$ and $\boldsymbol{\Omega}$, if $\boldsymbol{F}$ and $\boldsymbol{M}$ are known, or vice versa, and can be solved uniquely once the six basic flow fields $\boldsymbol{v}^j$, $j = 1, \ldots, 6$ and the resultant total forces and moments are computed. Equations (3.5)–(3.6) are the general Faxén law. These equations are equivalent to (6.9), (6.11) of Brenner (1964), or (46) of Brenner (1966), when dealing with infinite medium (no boundaries). Brenner (1966) also addresses the case of bounded or semi-bounded domains, pointing out the necessary modifications. His approach is equivalent to the use of the Green's function (2.13) for the appropriate region (see also Karrila & Kim 1989). The difference between this presentation and those of Brenner, is that Brenner did not have equation (2.11) to compute $\boldsymbol{f}(\boldsymbol{v}^j)$, $j = 1, \ldots, 6$, independently. He used expressions which amount to an expansion of $\bar{\boldsymbol{u}}$ in a Taylor series around the origin, and use of translational and rotational triadic 'stress' fields $\boldsymbol{\Pi}^t$ and $\boldsymbol{\Pi}^r$ (see e.g. Brenner & Haber 1983). Knowledge of these triadic fields is equivalent to knowledge of the six flow fields $\boldsymbol{v}^j$, $j = 1, \ldots, 6$.

For a particle in infinite medium, Kim (1985) and Kim & Karrila (1991), also bypass the difficulty of evaluating the integrals in (3.5), (3.6), by making what amounts to the following observation: the integral in (3.5) has the same functional form as equation (2.7) for the flow fields $\boldsymbol{v}^j$, with $\bar{\boldsymbol{u}}$ replacing the Stokeslets on the boundary,

$$v_k^j(\boldsymbol{y}) = \int t_{kl}(\boldsymbol{x}, \boldsymbol{y}) f_l(\boldsymbol{v}^j)\,\mathrm{d}S_x.$$

Thus, if we take $\boldsymbol{V} = \boldsymbol{\Omega} = 0$ (a stationary particle), the force F_j acting on it can be obtained by applying the same linear operator to $\bar{\boldsymbol{u}}$ as used to obtain $\boldsymbol{v}^j$ from the Stokeslets. In general, if

$$\boldsymbol{v}^j(\boldsymbol{y}) = \boldsymbol{e}^j \cdot \mathscr{F}\{\boldsymbol{t}(\boldsymbol{y} - \boldsymbol{x})\},$$

where $\boldsymbol{t}$ is the Stokeslet given in equation (2.6), $\mathscr{F}$ is a linear functional, and $\boldsymbol{x}$ the region over which the Stokeslets are distributed, then

$$F_j = \boldsymbol{e}^j \cdot \mathscr{F}\{\bar{\boldsymbol{u}}(\boldsymbol{x})\}.$$

This may be used to advantage if equivalent simpler expressions for $\mathscr{F}$ are known, as they demonstrate. A similar argument applies for the expression for the moments.

When looking at the force acting on a moving particle, (3.5), (3.6) give the correct way to add the additional contributions.

Notice that the above arguments hold for bounded regions as well, replacing $\boldsymbol{t}$ by $\boldsymbol{T}$, see (2.13), (2.14).

The above approach requires solving for the three translations and three rotations, that is, solving the resistance problem (prescribed velocities on the boundary). This can be done by first using the Power & Miranda (1987) technique, then using their expressions for the velocity and pressure, computing the stresses on this boundary, which can then be used in (3.5), (3.6).

Alternatively, one may choose to obtain six independent basic flows by prescribing the total forces and moments on the body, and solve the homogeneous equation (2.15), thus obtaining the stresses on the boundary directly.

Explicitly, define six legitimate fields $\boldsymbol{u}^i$, such that

$$\left.\begin{aligned} F_j(\boldsymbol{u}^i) = \delta_{ij}, \quad M_j(\boldsymbol{u}^i) = 0, \\ F_j(\boldsymbol{u}^{i+3}) = 0, \quad M_j(\boldsymbol{u}^{i+3}) = \delta_{ij} \end{aligned}\right\} \quad j = 1,2,3; \quad i = 1,2,3. \tag{3.7}$$

To obtain the Faxén laws using $\boldsymbol{u}^i$, we proceed as follows. Solving (2.15) for these six flow fields we compute $\boldsymbol{u}^i$ on ∂S using (2.14) with $\bar{\boldsymbol{u}} = 0$. Since

$$\boldsymbol{u}^i = \boldsymbol{V}^i + \boldsymbol{\Omega}^i \times \boldsymbol{r}, \quad i = 1, \ldots, 6, \tag{3.8}$$

we obtain $\boldsymbol{V}^i, \boldsymbol{\Omega}^i$, $i = 1, \ldots, 6$. As these six fields are linearly independent, we may write

$$\boldsymbol{v}^j = a_{j,i}\boldsymbol{u}^i, \quad j = 1, \ldots, 6, \tag{3.9}$$

with the Einstein summation convention, as before. To solve for $a_{j,i}$ the following linear system of equations is to be solved for each j:

$$\left.\begin{aligned} a_{j,i}\Omega^i_k = 0, \quad a_{j,i}V^i_k = \delta_{kj} \\ a_{j+3,i}\Omega^i_k = e^j_k, \quad a_{j+3,i}V^i_k = 0, \end{aligned}\right\} \quad k = 1,2,3; \quad j = 1,2,3. \tag{3.10}$$

In terms of $\boldsymbol{u}^i$, the equations for the velocity $\boldsymbol{V}$ and spin $\boldsymbol{\Omega}$ of the rigid body S then take the form:

$$\left.\begin{aligned} a_{j,l}V_l + a_{j,l+3}\Omega_l &= F_j(\boldsymbol{u}) + \int_{\partial S} \bar{u}_l a_{j,i} f_l(\boldsymbol{u}^i)\,\mathrm{d}S, \\ a_{j+3,l}V_l + a_{j+3,l+3}\Omega_l &= M_j(\boldsymbol{u}) + \int_{\partial S} \bar{u}_l a_{j+3,i} f_l(\boldsymbol{u}^i)\,\mathrm{d}S, \end{aligned}\right\} \quad j = 1,2,3. \tag{3.11}$$

Equations (3.11) are equivalent to (3.5), (3.6) relating the velocity and spin to the forces acting on the body. The classical Faxén laws follow immediately from the above results (see §4.1). Thus, once six basic flows are computed, $\boldsymbol{u}^j$ or $\boldsymbol{v}^j$, $j = 1, \ldots, 6$, the general Faxén laws can be used either for the problem of the first kind (the resistance problem), or the second kind (the mobility problem). Again, it should be emphasized that if one is interested in the resultant flow field and pressure field, one has to go back and solve the full equations (2.4), (2.5), (2.15).

An interesting point to note is the following. From a mathematical point of view, the resistance problem, for which velocities are given on the boundary, is well posed. On the other hand, replacing direct boundary conditions by indirect conditions – integrals over the boundary of derivatives of the solution – as in the mobility problem, raises the question of well posedness and uniqueness of the solutions. Physically, we expect these conditions to suffice. The existence of (2.11) or (2.15), with six independent homogeneous solutions, answers this problem in the affirmative. Once these are known, Faxén's laws (3.5), (3.6) (or (3.11)), show the equivalence of prescribing the velocity on the boundary (which for rigid-body motion means prescribing $\boldsymbol{V}$ and $\boldsymbol{\Omega}$), and prescribing the total force $\boldsymbol{F}$ and total moment $\boldsymbol{M}$. It also follows, therefore, that this is true for rigid-body motions only. Faxén's laws are, therefore, an intrinsic property of the equations, justifying the uniqueness and well posedness of the mobility problem.

4. Exact solutions

We shall show here how several known results, as well as new ones, follow directly from (2.7), (2.11) and the general Faxén law. For symmetric bodies moving along (or rotating around) one of their axes of symmetry, it was found (Happel & Brenner **1973**) that the only non-zero drag (or moment) is in the direction of motion. Thus, (3.5), (3.6) can be separated into components.

As was shown in §3, we need six independent solutions of the *homogeneous* part of (2.11) or (2.12).

4.1. *A Sphere in infinite medium* (*of radius a*)

4.1.1. *Translatory motion*

For translation in the x_m direction, $m = 1,2,3$, take $f_l(\boldsymbol{x}) = f\delta_{lm}$. It follows then that these are solutions of equation (2.11), provided

$$\int_{\partial S} \frac{r_i r_j}{r^3} \mathrm{d}S = \frac{4\pi a}{3}\delta_{ij},$$

which is easily confirmed. Thus, f is constant and has a component in the direction of motion only, so that $f = F/4\pi a^2$. Inserting this into (2.7), evaluating at the origin (which is permissible since (2.7) holds inside the body as well), we have,

$$u_k(\mathbf{0}) = \frac{F}{4\pi a^2}\delta_{lm}\int_{\partial S} t_{lk}(\boldsymbol{x},\mathbf{0})\,\mathrm{d}S_x = \frac{F}{4\pi a^2}\delta_{lm}\frac{2a}{3\mu}\delta_{lk} = \frac{F}{6\pi\mu a}\delta_{mk}, \quad m = 1,2,3,$$

which is Stokes' law (evaluation of the integral being trivial).

To obtain Faxén's first law for a sphere, take (3.5) for the above case with $F = 6\pi\mu a\,(\boldsymbol{u} = \mathbf{1})$. Then,

$$F_m(\boldsymbol{u}) = V_m\, 6\pi\mu a - \int_{\partial S} \bar{u}_m \frac{6\pi\mu a}{4\pi a^2}\mathrm{d}S = 6\pi\mu a\left\{V_m - \frac{1}{4\pi a^2}\int_{\partial S}\bar{u}_m\,\mathrm{d}S\right\}.$$

But
$$\frac{1}{4\pi a^2}\int_{\partial S}\bar{u}_m\,\mathrm{d}s = \bar{u}_m(0) + \frac{a^2}{6}\nabla^2\bar{u}_m(0)$$

(see Brenner 1964; Caflisch & Rubinstein 1986), and Faxén's first law results.

4.1.2. *Rotation*

For a sphere rotating around a given axis, the equations again separate. For rotation around the y_3 axis (coordinates measured from the centre of the sphere), for example, one easily confirms that if

$$f(\boldsymbol{y}) = \frac{3\mu}{a}(-y_2, y_1, 0)\,\Omega, \tag{4.1}$$

then for the moment, $M_j = 8\pi\mu a^3\,\Omega\delta_{j3}$, and, using (2.7) on the surface of the sphere,

$$\boldsymbol{u} = \Omega(-y_2, y_1, 0) = \Omega\boldsymbol{e}_3\times\boldsymbol{r}, \tag{4.2}$$

with similar results for the other directions. Now, using (3.6), one obtains

$$M_j = 8\pi\mu a^3 \Omega_j - \int_{\partial S} \bar{\boldsymbol{u}}\cdot\boldsymbol{f}(\boldsymbol{v}^{j+3})\,\mathrm{d}S,$$

$$= 8\pi\mu a^3 \Omega_j - 12\pi\mu a \boldsymbol{e}_j\cdot\left(\frac{1}{4\pi a^2}\int_{\partial S} \boldsymbol{r}\times\bar{\boldsymbol{u}}\,\mathrm{d}S\right),$$

and a simple calculation yields

$$M_j = 8\pi\mu a^3\Omega_j - 4\pi\mu a^3(\nabla\times\bar{\boldsymbol{u}}(\mathbf{0}))_j, \tag{4.3}$$

which is Faxén's second law.

We see that the solution for a sphere is particularly appealing. To obtain a constant velocity we distribute constant stress, and to obtain rotation ($\boldsymbol{e}_3\times\boldsymbol{r}$) we distribute 'rotation'.

4.2. *Motion of an ellipsoid*

The general case of an ellipsoid translating in a quiescent fluid was solved by Oberbeck (Lamb 1945, p. 604), see below. The case of a rotating ellipsoid was first solved by Edwardes (1892), using ellipsoidal harmonics. Later Jeffrey (1922) treated motion of ellipsoids in uniform shear flow. Brenner (1964, 1966) treated this problem again when dealing with Faxén laws (see also Happel & Brenner 1973), and again in Brenner & Condiff (1974), and in Brenner & Haber (1983).

The motion of a spheroid of revolution has also received extensive treatment in the literature. Chwang & Wu (1974) treated the case of rotation around the major axis by distributing Rotlets on the axis between the foci (with a parabolic distribution of strengths). Later, Chwang & Wu (1975) treated translatory motion and rotation around the minor axis. Chwang (1975) solved for a prolate spheroid for several external flow fields, eventually enabling him to evaluate the velocity and spin of the spheroid, at arbitrary orientation and position, in a paraboloid flow.

The detailed solutions of Oberbeck and Edwardes were utilized by Brenner (1966) to obtain a symbolic operator expression for the Faxén law. Brenner & Haber (1983) show in the case treated by Chwang (1975) (quadratic flows) that the symbolic operator method of Brenner truncates from an infinite series to finite (simple) expressions, from which Chwang's results 'follow at once'. Chwang's results are easy to reproduce by our method as well once the six independent solutions for the homogeneous equations (2.11) or (2.12) are known.

Kim (1985) and Kim & Karrila (1991) use the form of the Faxén law that they developed to deal with spheroids, utilizing the Chwang & Wu (1975) solutions.

4.2.1. *Spheroid of revolution – translatory motion*

Consider a prolate spheroid defined by

$$\frac{x_1^2}{a^2}+\frac{x_2^2+x_3^2}{b^2} = 1, \quad c^2 = a^2-b^2 = e^2a^2. \tag{4.4}$$

For any two points $\boldsymbol{x},\boldsymbol{y}$ on the surface of the spheroid, we have

$$[a^2-e^2x_1^2]^{\frac{1}{2}}\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{r} = -[a^2-e^2y_1^2]^{\frac{1}{2}}\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{r},$$

where $\boldsymbol{r}$ is the radius vector connecting points $\boldsymbol{x}$ and $\boldsymbol{y}$. Using the above relation, one easily obtains that the homogeneous solution of (2.11) is

$$f_i(\boldsymbol{y}) = \frac{F_j}{4\pi b}[a^2-(ey_1)^2]^{-\frac{1}{2}}\delta_{ij}, \quad i,j = 1,2,3, \tag{4.5}$$

where $\boldsymbol{F}$ is the total force. To obtain the relation between the force F_j and the translatory velocity u_j, use (2.7). A convenient point at which to calculate is the origin, which results in a couple of simple integrals and the result is

$$\left.\begin{aligned} u_1 &= \frac{F_1}{16\pi\mu}\frac{1}{ae^3}\left[(e^2+1)\ln\frac{1+e}{1-e}-2e\right], \\ u_{2,3} &= \frac{F_{2,3}}{32\pi\mu}\frac{1}{ae^3}\left[(3e^2-1)\ln\frac{1+e}{1-e}+2e\right], \end{aligned}\right\} \tag{4.6}$$

as obtained by Chwang & Wu (1975), after some lengthy calculations.

4.2.2. *Spheroid of revolution – rotation*

For the three rotations we obtain, similarly to the previous case, that the solution to (2.11) ($\bar{\boldsymbol{f}}=0$) is,

$$\boldsymbol{f}(\boldsymbol{y}) = f_j^*(a^2-e^2y_1^2)^{-\frac{1}{2}}\boldsymbol{e}_j\times\boldsymbol{r}, \tag{4.7}$$

which can be checked directly by substitution into (2.11) and computing the necessary integrals. f_j^* is related to M_j via

$$f_j^* = \frac{3M_j}{8\pi b^3}\delta_{j1}+\frac{3M_j}{4\pi b(a^2+b^2)}(\delta_{j2}+\delta_{j3}). \tag{4.8}$$

On the surface of the spheroid the velocity is

$$\boldsymbol{u} = \Omega_j\boldsymbol{e}_j\times\boldsymbol{r}, \tag{4.9}$$

where

$$\Omega_j = \frac{3}{32}\frac{M_j}{\mu\pi ab^2}\frac{1}{e^3}\left[2e-(1-e^2)\ln\frac{1+e}{1-e}\right]\delta_{j1} + \frac{3}{32}\frac{M_j}{\pi\mu a(a^2+b^2)e^3}\left[-2e+(1-e^2)\ln\frac{1+e}{1-e}\right](\delta_{j2}+\delta_{j3}). \tag{4.10}$$

The above results follow from (2.4), (2.5), (2.7).

4.2.3. *A spheroid in a parabolic flow*

Chwang (1975) solved the problem of a spheroid in free motion in a parabolic flow, $\boldsymbol{u} = (\eta^2+\zeta^2)\boldsymbol{e}_\xi$, (see figure 1) at an arbitrary location and orientation with respect to the impinging flow. Since the impinging flow has a non-zero component in the ξ-direction only, it follows from symmetry considerations that the only component of velocity of the ellipsoidal centre is also in the direction of ξ.

Insert this result into (3.5), adding the condition $F_j(\boldsymbol{u})=0$, for free motion, to obtain

$$F_\xi(\boldsymbol{v}_\xi)V_\xi = \int_{\partial S}\boldsymbol{u}\cdot\boldsymbol{f}(\boldsymbol{v}_\xi)\,\mathrm{d}S, \tag{4.11}$$

where $\boldsymbol{v}_\xi$ is the basic flow field for translation in the ξ-direction. From (4.5) $F_\xi(\boldsymbol{v}_\xi)$ is related to $\boldsymbol{f}(\boldsymbol{v}_\xi)$ via

$$f_\xi(\boldsymbol{v}_\xi) = F_\xi(\boldsymbol{v}_\xi)/4\pi ab(1-e^2x^2)^{\frac{1}{2}}.$$

Substituting this into (4.11) we obtain

$$V_\xi = \frac{1}{4\pi ab}\int_{\partial S}(\eta^2+\xi^2)(1-e^2x^2)^{-\frac{1}{2}}\,\mathrm{d}S.$$

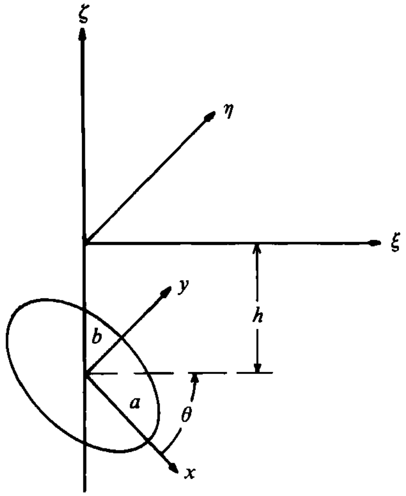

FIGURE 1. A spheroid at an arbitrary orientation with respect to the direction of flow.

Using ellipsoidal coordinates we end up with a simple integral, to obtain

$$V_\xi = h^2 + \frac{a^2}{3}(2 - e^2 - e^2\cos^2\theta),$$

as in Chwang (1975).

In a similar manner, use (3.6) to obtain

$$\Omega_z M_z(\boldsymbol{v}^6) = \int_{\partial S} \boldsymbol{u}\cdot\boldsymbol{f}(\boldsymbol{v}^6)\,\mathrm{d}S,$$

where, by (4.8),
$$f_z(\boldsymbol{v}^6) = \frac{3M_z(\boldsymbol{v}^6)}{4\pi b(a^2+b^2)}\delta_{j3}.$$

yielding a definite integral for Ω_z. This again turns out to be a simple integral, in ellipsoidal coordinates, and results in

$$\Omega_z = 2h(1 - e^2\cos^2\theta)/(2 - e^2),$$

as in Chwang (1975).

4.2.4. *Translation of a general ellipsoid*

Let the ellipsoid be

$$\phi(x_1, x_2, x_3) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1. \tag{4.12}$$

The Oberbeck solution is expressed in terms of derivatives of the integrals:

$$\Omega = \pi a_1 a_2 a_3 \int_\lambda^\infty \left(\frac{x_1^2}{a_1^2+s} + \frac{x_2^2}{a_2^2+s} + \frac{x_3^2}{a_3^2+s} - 1\right)\frac{\mathrm{d}S}{\Delta(s)} = \int_\lambda^\infty g(x_1, x_2, x_3, s)\frac{\mathrm{d}s}{\Delta(s)},$$

$$\chi = a_1 a_2 a_3 \int_\lambda^\infty \frac{\mathrm{d}s}{\Delta(s)},$$

where $\Delta(s) = [(a_1^2+s)(a_2^2+s)(a_3^2+s)]^{\frac{1}{2}}$, and λ is the positive root of the cubic equation (in s) $g(x_1, x_2, x_3, s) = 0$. We shall give a different representation.

Let $D(\boldsymbol{x}) = |\nabla\phi(\boldsymbol{x})|$, then for any two points $\boldsymbol{x},\boldsymbol{y}$ on the ellipsoid we have

$$D(\boldsymbol{y})\boldsymbol{r}_{yx}\cdot\boldsymbol{n}(\boldsymbol{y}) = \boldsymbol{r}_{yx}\cdot\nabla\phi(\boldsymbol{y}) = (x_i-y_i)\frac{2y_i}{a_i^2} = 2\left(\frac{x_i y_i}{a_i^2}-1\right) = -D(\boldsymbol{x})\boldsymbol{r}_{yx}\cdot\boldsymbol{n}(\boldsymbol{x}).$$

Define
$$f_i(\boldsymbol{x}) = f_i^*/D(\boldsymbol{x}), \quad i = 1,2,3, \tag{4.13}$$

then (for $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}$)

$$\int_{\partial S}\frac{r_i r_j r_k f_k(\boldsymbol{x})}{r^5}\,\mathrm{d}S_x n_j(\boldsymbol{y}) = -\int_{\partial S}\frac{r_k f_k^* r_i r_j n_j(\boldsymbol{x})}{D(\boldsymbol{y})\,r^5}\,\mathrm{d}S_x = \frac{2\pi}{3}\frac{f_i^*}{D(\boldsymbol{y})}$$

(see Ladyzhenskaya 1963), so (4.13) satisfies (2.11). By (2.7) we therefore have the solution as

$$u_k(\boldsymbol{y}) = f_l^*\int_{\partial S} t_{lk}(\boldsymbol{x},\boldsymbol{y})/D(\boldsymbol{x})\,\mathrm{d}S_x, \tag{4.14}$$

and for any point $\boldsymbol{y}$ on the boundary, the only contribution to U_k comes from the $k-k$ component of t_{lk},

$$U_k = f_l^*\,\delta_{mk}\int_{\partial S} t_{lm}(\boldsymbol{x},\boldsymbol{y})/D(\boldsymbol{x})\,\mathrm{d}S_x. \tag{4.15}$$

The total force is

$$F_l = f_l^*\int_{\partial S}\frac{\mathrm{d}S_x}{D(\boldsymbol{x})}, \quad l = 1,2,3. \tag{4.16}$$

For the relation between the total force and the translation velocity, see also Happel & Brenner (1973), or Kim & Karrila (1991).

4.2.5. *Rotation of a general ellipsoid*

In a straightforward manner one may check that the solution for the three rotations is

$$\boldsymbol{f}(\boldsymbol{x}) = f_j^* D^{-1}(\boldsymbol{x})\,\boldsymbol{e}_j\times\boldsymbol{r}, \tag{4.17}$$

and, using (2.7), we have an explicit expression in terms of a finite surface integral, for the solution in this case. Expressions for the relation between f_j^*, the moments and the rotational velocity, are straightforward using (2.5) but not explicit (expressed as finite integrals). Alternative expressions for the relation between the moment and the rotational velocity may be deduced from Brenner (1964) or Kim & Karrila (1991).

5. Numerical solutions

In this section we will demonstrate several numerical solutions, both for rigid bodies in infinite medium and for bodies near boundaries, and compare results with known solutions. We use a simple numerical scheme to solve the singular equation. This scheme, though not highly accurate, suffices to demonstrate the power of (2.11) or (2.12). We first present results for a slender torus in symmetric motion, where the first stages of the solution are analytic. We then present the solution scheme and compute motions of ellipsoids and tori in infinite medium. To demonstrate applications of (2.15) we solve for a sphere falling onto a plane boundary, which was

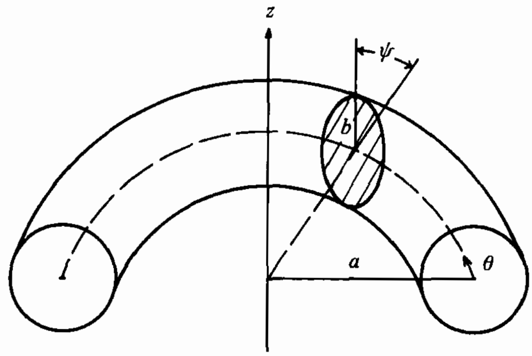

FIGURE 2. A torus with its system of coordinates.

solved originally by Brenner (1961). For a more complicated case we solve for a sphere falling towards one wall between parallel plane walls. This problem was the subject of the paper by Ganatos *et al.* (1980*a*).

5.1. *Semi-numerical solutions*: *torus*

5.1.1. *Torus translating along an axis of symmetry*

Consider a torus with axes a, b (see figure 2), moving along the z-axis. Symmetry considerations imply the following form for the stress function, given in body coordinates:

$$\boldsymbol{f}(\theta,\psi) = (g(\psi)\cos\theta, g(\psi)\sin\theta, f(\psi)). \tag{5.1}$$

Because of symmetry, one can integrate analytically in the θ-direction obtaining a pair of equations for the coefficients f, g in (5.1),

$$\begin{aligned} f(\psi_1) = {} & \frac{3b}{\pi(2(a+b\cos\psi_1))^{\frac{3}{2}}}\int_0^{2\pi}\frac{(\sin\psi_1-\sin\psi)\,(a+b\cos\psi)^{\frac{1}{2}}}{(1-\cos(\psi_1-\psi))\,(2+\alpha)^{\frac{1}{2}}} \\ & \times\{f(\psi)\,(\sin\psi_1-\sin\psi)\,S-g(\psi)[(\cos\psi-\cos\psi_1-b\beta)\,S+\beta P]\}\,\mathrm{d}\psi, \end{aligned} \tag{5.2a}$$

$$\begin{aligned} g(\psi_1) = {} & \frac{3}{\pi(2(a+b\cos\psi_1))^{\frac{3}{2}}}\int_0^{2\pi}\left(\frac{a+b\cos\psi}{2+\alpha}\right)^{\frac{1}{2}}\left\{f(\psi)\,[(\cos\psi_1-\cos\psi-b\beta_1)\,S+\beta_1 P]\right. \\ & \times\frac{\sin\psi_1-\sin\psi}{1-\cos(\psi_1-\psi)}+g(\psi)\left[\left(2+\alpha-\frac{(\sin\psi_1-\sin\psi)^2}{1-\cos(\psi-\psi_1)}-b^2\gamma\right)S\right. \\ & \left.\left.+ab\gamma E(\tfrac{1}{2}\pi,\delta)+\left(b^2\cos\psi_1\gamma-\frac{a}{b}\alpha\right)F(\tfrac{1}{2}\pi,\delta)-(2+\alpha)\cos\psi_1 E(\tfrac{1}{2}\pi,\delta)\right]\right\}\mathrm{d}\psi, \end{aligned} \tag{5.2b}$$

$$\left.\begin{aligned} & \alpha = \alpha(\psi,\psi_1) = \frac{b^2(1-\cos(\psi-\psi_1))}{(a+b\cos\psi)\,(a+b\cos\psi_1)}; \quad \beta = \beta(\psi,\psi_1) = \frac{1-\cos(\psi-\psi_1)}{a+b\cos\psi}; \\ & \beta_1 = \beta(\psi_1,\psi), \quad \gamma = \gamma(\psi,\psi_1) = \frac{(\sin\psi-\sin\psi_1)^2}{(a+b\cos\psi)\,(a+b\cos\psi_1)}, \\ & \delta = (1+\tfrac{1}{2}\alpha)^{-\frac{1}{2}}, \quad P = \alpha E(\tfrac{1}{2}\pi,\delta)+b\cos\psi_1 F(\tfrac{1}{2}\pi,\delta), \\ & S = E(\tfrac{1}{2}\pi,\delta)\cos\psi_1+\frac{a}{3b(2+\alpha)}[4(1+\alpha)\,E(\tfrac{1}{2}\pi,\delta)-\alpha F(\tfrac{1}{2}\pi,\delta)]. \end{aligned}\right\} \tag{5.2c}$$

Here, E and F denote the usual elliptic integrals. The equations are now regular, with the values at $\psi = \psi_1$ of the integrand in (5.2*a*) and (5.2*b*) being $A\cos\psi_1$, $-A\sin\psi_1$, respectively. Here

$$A = [2(a+b\cos\psi_1)]^{\frac{1}{2}}\left(\cos\psi_1+\frac{2a}{3b}\right)(f(\psi_1)\cos\psi_1-g(\psi_1)\sin\psi_1).$$

Since the equations are now regular, it is a trivial matter to solve them numerically. Using (2.7), one can obtain a relation between the total force F and the velocity U:

$$U = \frac{\sqrt{2}b}{8\pi\mu}F\int_0^{2\pi}\frac{a+b\cos\psi}{[2a(a+b\cos\psi)+b^2(1-\sin\psi)]^{\frac{1}{2}}}\left\{2\left(f(\psi)-\frac{b(1-\sin\psi)}{a+b\cos\psi}g(\psi)\right)F(\tfrac{1}{2}\pi,\delta)\right.$$
$$\left.+\left[(1-\sin\psi)\left(f(\psi)-\frac{bg(\psi)\sin\psi}{a+b\cos\psi}\right)-\frac{ag(\psi)\cos\psi}{a+b\cos\psi}\right]E(\tfrac{1}{2}\pi,\delta)\right\}\mathrm{d}\psi, \quad (5.3)$$

where δ is defined in (5.2*c*).

It is interesting to compare this result with the expression for a slender torus ($b \ll a$) obtained by Johnson & Wu (1979),

$$U = \frac{F}{a\pi\mu}\left(\frac{1}{2}+\ln\frac{8a}{b}\right). \quad (5.4)$$

As b/a decreases from 0.2 to 0.05, the agreement goes up from 97.4 to 99.5%.

5.1.2. *Torus rotating around an axis of symmetry*

Consider the above torus (see figure 2) rotating around the z-axis. Here it can be shown that the stress has the following functional form:

$$f(\theta,\psi) = (-f(\psi)\sin\theta, f(\psi)\cos\theta, 0), \quad (5.5)$$

and, after integrating through θ, one obtains,

$$f(\psi_1) = \frac{3b}{\sqrt{2}\pi(a+b\cos\psi_1)^2}\int_0^{2\pi}\left[\frac{(a+b\cos\psi)(a+b\cos\psi_1)}{2+\alpha}\right]^{\frac{1}{2}}$$
$$\times\left\{\frac{a}{3b}((1+\alpha)E(\tfrac{1}{2}\pi,\delta)-\alpha F(\tfrac{1}{2}\pi,\delta))+((1+\alpha)F(\tfrac{1}{2}\pi,\delta)-(2+\alpha)E(\tfrac{1}{2}\pi,\delta))\cos\psi_1\right\}f(\psi)\,\mathrm{d}\psi, \quad (5.6)$$

where α, δ are defined in (5.2*c*). Equation (5.6), together with the specification of the moment,

$$M = 2\pi b\int_0^{2\pi}(a+b\cos\psi)^2 f(\psi)\,\mathrm{d}\psi, \quad (5.7)$$

is then solved. Some values of the stress coefficient functions $f(\psi)$ for several values of the aspect ratio, $\epsilon = b/a$, are shown in figure 3. Again, it is interesting to compare this result with Johnson & Wu (1979) for a slender rotating torus. For the comparison we compute u_θ at $\psi = \frac{1}{2}\pi$, and obtain

$$\left.\begin{aligned}u_\theta &= \frac{-Mb}{\sqrt{2}\pi\mu a}\int_0^{2\pi}[a(a+b\cos\psi)/(2+\alpha)]^{\frac{1}{2}}\{(1+\alpha)F(\tfrac{1}{2}\pi,\delta)-(2+\alpha)E(\tfrac{1}{2}\pi,\delta)\}f(\psi)\,\mathrm{d}\psi,\\ \alpha &= b^2(1-\sin\psi)/[a(a+b\cos\psi)], \quad \delta = (1+\tfrac{1}{2}\alpha)^{-\frac{1}{2}},\end{aligned}\right\} \quad (5.8)$$

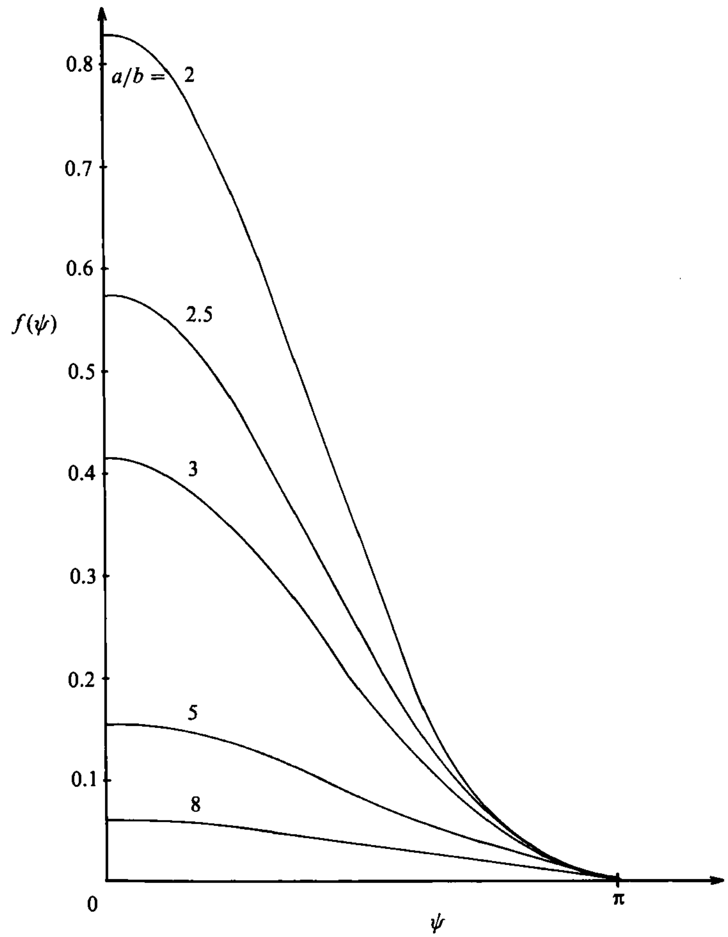

FIGURE 3. Values of the stress function $f(\psi)$ for several values of $\epsilon = b/a$, for a rotating torus (see (5.5)–(5.6)).

whereas Johnson & Wu obtain

$$u_\theta = -\frac{M}{4\pi^2 \mu a}\left(\ln \frac{8a}{b} - 2\right). \tag{5.9}$$

Comparing the two expressions for several values of the aspect ratio $\epsilon = b/a$ shows the slender-body approximation to be excellent only for smaller values of ϵ, with a deviation of 10% at $\epsilon = 0.2$, down to 0.5% at $\epsilon = 1/60$.

5.2. *General numerical procedure*

For a body of arbitrary shape, a numerical method has to be applied to solve (2.4), (2.5), (2.15). Owing to the singularity of the integrands a simple algorithm is not adaptable here. Power & Miranda (1987) have similar singularities in their integral equations. They claim that using an equation such as (2.12) instead of (2.11) leads to cancellation of the integrand at the singular point. As pointed out in §2.1, this is correct only if one works, for example, with the differential of the solid angle (and not with dS, the surface differential, as they in fact, incorrectly, did). One possibility is

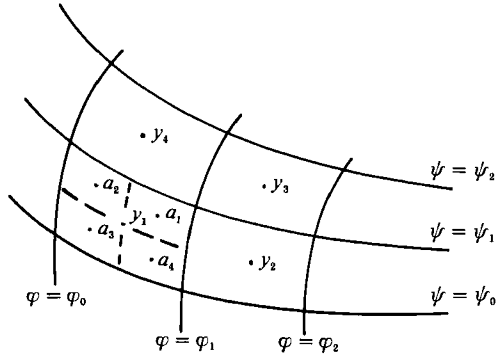

FIGURE 4. The numerical mesh used for the integral equation (2.15), at the point $\boldsymbol{y} = \boldsymbol{y}_1$.

to use a Galerkin method (Gavze 1989), but the problem of choosing surface elements still remains. We choose to solve the equations by a simple and straightforward method. The method is not very accurate, but suffices to demonstrate the appeal of (2.15). The surface of the body is partitioned into elements, and the value of the integrand at each element is represented by its value at the centre. This holds everywhere except at the singular point, where we use a finer partition. The element that contains the singularity is partitioned into four sub-elements, as shown in figure 4.

A bi-linear interpolation is used on the stresses in the sub-elements. Thus, for example,

$$f(a_1) = \tfrac{1}{16}(9f(y_1) + f(y_3) + 3f(y_2) + 3f(y_4)) \tag{5.10}$$

(see figure 4). The numerical method is sensitive to the partition. A partition that conserves the symmetry of a problem yields better results. Also, better results are obtained if a partition is chosen such that for any two centre points $\boldsymbol{x}$ and $\boldsymbol{y}$, all three components of $\boldsymbol{x}-\boldsymbol{y}$ are similar in size. Using the numerical scheme we checked the numerical solution against known solutions:

A sphere in translatory motion. The solution in this case is $\boldsymbol{f}=$ const. (see §4.1.1). A numerical solution with 144 elements gave an error of less than 1%. 64 elements yields a solution with less than a 2% error.

Translatory motion of an ellipsoid. An analytic solution is available (see §4.2.1). For 100 elements and aspect ratio of 1:2 a maximal error of less than 5% is achieved for f_x, the only component of force one should obtain. The other two components of force should vanish. We obtain for them $\max|f_y, f_z| = 0.22$ compared to a typical value of 4.0–8.0 for f_x.

For an ellipsoid with aspect ratio of 1:5, using 144 elements we again obtain an error of about 5% for f_x, except near the leading edge where the error grows up to 10%. f_x now varies between 1.5 and 7.5, and again $\max|f_y, f_z| = 0.2$.

Motion of a torus. For the two motions in §5.1 above, the results are accurate and can be used for comparison. In both motions, for various aspect ratios, errors did not exceed 10% with better accuracy at many points. Exceptional points of lower accuracy were seen where the local value of f was much smaller than the maximal value of f.

5.3. *A body moving next to an infinite flat plate*

For a body moving next to a plane, (2.15) is used where the Green's function for a Stokeslet above a flat plate is to be used (Blake 1971).

h	λ (Brenner)	λ (Present)	Ratio %
1.54	0.01747	0.01740	99.6
2.35	0.0289	0.02876	99.5
3.76	0.03755	0.03726	99.2
6.13	0.0434	0.04302	99.1
10.068	0.04715	0.04671	99.1

TABLE 1. Comparison of Brenner's (1961) results and ours for the force velocity relation (see (5.12)), for a unit sphere moving towards a plane at various heights h

Let $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y} = (x_1-y_1, x_2-y_2, x_3-y_3)$, and let $\boldsymbol{R} = (x_1-y_1, x_2-y_2, x_3+y_3)$ be the radius vector from the reflected point (in the plane $x_3 = 0$). The parallel version of (2.12) to (2.15) now becomes

$$f_i(\boldsymbol{y}) = \bar{f}_i(y) + \frac{3}{4\pi}\int_{\partial S} \frac{r_i r_j r_k}{r^5} f_k(\boldsymbol{x})\,(n_j(\boldsymbol{y})-n_j(\boldsymbol{x}))\,\mathrm{d}S_x$$
$$+\frac{3}{4\pi}\int_{\partial S}\frac{r_i r_j r_k}{r^5} n_j(\boldsymbol{x})\,(f_k(\boldsymbol{x})-f_k(\boldsymbol{y}))\,\mathrm{d}S_x - \frac{3}{4\pi}\int_{\partial S}\frac{R_i R_j R_k}{R^5} f_k(\boldsymbol{x})\,\mathrm{d}S_x\, n_j(\boldsymbol{y})$$
$$-\frac{3}{2\pi}\int_{\partial S}\frac{x_3}{R^5}\left\{-R_i R_j f_3(\boldsymbol{x}) - \delta_{ij} x_3 R_k f_k(\boldsymbol{x}) + y_3(R_j f_i(x) + R_i f_j(\boldsymbol{x}))\right.$$
$$\left. -2y_3(R_i\delta_{j3}+R_j\delta_{i3}) f_3(\boldsymbol{x}) + 2\delta_{ij} x_3 R_3 f_3(\boldsymbol{x}) - \frac{5y_3 R_i R_j (R_k f_k - 2R_3 f_3)}{R^2}\right\}\mathrm{d}S_x\, n_j(\boldsymbol{y}). \tag{5.11}$$

As an example, consider a unit sphere moving towards a plane wall. Brenner (1961) solved this problem, and expressed the relation between the force and velocity as

$$F = 6\pi\mu U\lambda, \tag{5.12}$$

where λ was given as an infinite series. From symmetry considerations one easily sees that the stress can be expressed as

$$f = (k(\eta,h)\cos\varphi, k(\eta,h)\sin\varphi, g(\eta,h)) \equiv (k(\eta,h), 0, g(\eta,h)), \tag{5.13}$$

expressed in Cartesian and cylindrical coordinates, respectively. Here η, φ are the longitudinal and circumferential angles, respectively, and h is the height of the sphere's centre above the plane. A comparison between our results with 64 elements and Brenner's results shows excellent agreement, as is shown in table 1.

5.4. *A body moving between two parallel flat plates*

As a final example, consider the motion of a body between two parallel flat plates. Ganatos *et al.* (1980*a*) solved for a sphere moving perpendicular to a plane wall. The method developed in the paper applies to a sphere only. On the other hand, Liron & Mochon (1976) solved for a Stokeslet between parallel plates. Using this solution one may apply (2.15) to any body. As a test case we compare results of our computations with those of Gantos *et al.* (1980*a*) in table 2. Our results show very nice accuracy even though, as mentioned in the beginning, we use a rather crude numerical model.

h	H	λ (Ganatos *et al.*)/ no. of elements	λ (present)/ no. of elements	Error %
5	50.3	1.296/16	1.32/91	<2
5	10	1.397/16	1.423/100	2
2	4	2.789/36	2.868/96	<3
2	10	2.135/36	2.202/96	3
1.5	3	4.78/100	4.948/96	3.5

TABLE 2. Comparison of Ganatos *et al.*'s (1980*a*) results and ours, for the force velocity relation (see (5.12)) of a unit sphere moving between, and towards, parallel plane walls at various heights h above one plate, and distance H between the plates

6. Discussion

We have presented here a singular integral equation of the second kind for the stresses on a rigid particle in Stokes flow. The integral equation is the natural one to consider, since the total force and total moment, and not velocities, are usually given. We have also demonstrated the generality of the method and the relative simplicity in the use of the equation, due to the simple representation of the flow field given in (2.14). The use of the general Faxén law implies that if one is only interested in the motion of the particle, it is sufficient to find six independent solutions of the homogeneous equation (2.15), corresponding essentially to the three independent translations and rotations. Of course, if we are not dealing with motion in an infinite medium, these solutions will depend on the distance and orientation, with respect to the boundaries. Thus, it may be necessary to recompute them every time step when following the motion of a particle.

Equation (2.15) is a singular integral equation, but it has a weak singularity. Nevertheless a good numerical scheme is desirable to overcome the singularity problem. We used a very simple and naïve scheme which, for the problems discussed, showed surprisingly accurate results. It is clear, however, that the method is not sufficiently accurate to properly solve for particles with less symmetry and smoothness.

The method described works for any particle moving next to boundaries, using (2.14) and (2.15). Thus, we have removed the restriction of having a 'nice' particle in order to obtain a solution. But the Green's function for the region has to be known, and this is only available for a limited number of regions with 'nice' boundaries, such as one plane wall, the region between parallel plane walls, or inside a straight circular cylinder. For more general regions, assume that a Green's function T_{lk} can be used which vanishes only on part of the stationary boundary. Then, (2.14) and (2.15) still hold, except that now the integral is over the surface of the particle *and* over the part of the boundary where T_{lk} does not vanish. This may result in an infinite integral. We have not checked such cases, and we do not know if, and for what cases, such an approach is feasible.

Recently, Karrila & Kim (1989) and Karrila *et al.* (1989) developed an approach that also lead to a Fredholm integral equation of the second kind for the density of the double layer. The method they proposed involves the simultaneous solution for both the unknown surface density and the translation and spin velocities for the rigid body. On the other hand the surface density is non-physical. They have an additional constraint (as part of the solution) in that the surface density should be orthogonal to the surface densities of the rigid-body motions. Our approach, on the other hand solves for the physical surface density (stresses) directly, and then the forces and moments are computed using the general Faxén laws, which involves direct integration over a surface, and solution of a system of linear equations. For particles in infinite medium our kernel is the adjoint of the kernel in the work of Karrila *et al.* As mentioned above, their method requires knowledge of the surface densities of rigid-body motions which are solutions of their homogeneous integral equation. These solutions are known for arbitrarily shaped bodies, and for multiparticle systems they remain the same (by adding zero density on all other particles, one at a time, see Karrila *et al.*, p. 920). Thus, the bulk of the work is in the solution of the extended integral equations. In our method, if the surface densities of the rigid-body motions were known then, by the generalized Faxén laws, one could easily obtain velocities given forces and moments, or vice versa. These densities are solutions of our homogeneous integral equation, and there are no simple solutions for arbitrarily shaped bodies. Thus, the bulk of the work is in finding the homogeneous solutions. For regions with boundaries, we utilize the appropriate Green's function for the region, if known, without much additional work. Using the Green's function for the region in the Karrila *et al.* approach destroys the advantage of the pre-knowledge of the rigid-body motion solutions. They are therefore forced to add a double-layer density function on the boundary, and an equation for the boundary as well, resulting in increased work. It would be interesting in future work to compare results of the two methods using similar numerical techniques.

This research was supported by the Technion V.P.R. Fund – K. & M. Bank Mathematics Research Fund, and by the Fund for the Promotion of Research at the Technion.

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